From symmetry group to free field equations

B. Sazdović

Institute of Physics Belgrade

Dragan Popović Memorial Conference 2025

I am honored to participate in this Dragan Popović Memorial Conference. Dragan was my colleague and friend. We published several papers together.
As far as I remember, for a while Dragan gave lectures in QFT. Recently I investigated new approach to classical field theory. Today I will talk about this approach.

Special theory of relativity

- Quantum field theory is one of most important theories in physucs. It combines the quantum theory, the field concept and the principle of special relativity.
  Since we will be working here with classical free theory we will pay attention to Special theory of relativity
- Special theory of relativity is a theory in four dimensional Minkowski space-time with three space directions $\vec{x} = \{x, y, z\}$ and one time direction t.
- Lline element in Minkowski space-time is

$$dx^2 = dt^2 - d\vec{x}^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (1)$$

  It is a slightly modified Pythagorean theorem, up to sign. Lline element is not always positive. The zero value of lline element define light cone.
- Symmetries of Minkowski space-time
  Poincare transformations are all four dimensional rotations (space rotations and boosts) and translations which preserve lline element. Generators of these transformations, translations P_a and rotations M_{ab}, form a Poincare algebra.

Quantum field theory

- In traditional approach quantum field theory starts with with well known equations
  Klein-Gordn $s = 0$,
  Dirac $s = \frac{1}{2}$,
  Maxwell $s = 1$,
  Rarita-Schwinger $s = \frac{3}{2}$,
  Einstein $s = 2$, ...
  equations $s > 2$
- Is it possible to unify all these equations with arbitrary spin? The answer is YES.
- How many equations do we need to describe all the equations of free field theory?
  - Physicists would say it depends on the number of particle properties associated with space-time symmetry.
  - Mathematicians would say that it depends on the symmetry group, which means on the number of Casinmir operators

Number of Principle field equations for Poincare group

- Ansver for Poincare group
- For massive particles in both cases the answer is TWO. There are two particle properties mass m and spin s and there are two Poincare Casinmir operators. In fact mass and spin are eigenvalues of Poincare Casinmir operators.
- For massless particles in both cases the answer is ONE. There is one particle property helicity λ and there is one Poincare Casinmir operator.
- Covariant description of helicity needs three independent equations. One is enough to describe helicity. What is the role of the other two equations? They are the source of local gauge invariance, well known for massless fields.
- Lorentz transformations are source of local gauge transformations for massless fields

Method

- In order to describe field theory we will prefer Weinberg approach who uses Wigner's definition of particles as irreducible representations of Poincare group .
- Unlike Weinberg approach which starts with particles and get to the field equations later we will start with principle field equations for arbitrary spin and show that all known equations for free fields follow from our principal equations.
- We offer general prescription which can be used to construct field theory based on symmetry of some given group. It means that all theories with same symmetry grpup will look the same at sufficiently low energy.
- Here we will apply this prescription to the Poincare group.

Poincare algebra

- Lie algebra of Poincare group has a form

$$[P_a, P_b] = 0\,, \qquad [P_c, M_{ab}] = -i\Big(\eta_{bc} P_a - \eta_{ac} P_b\Big)\,, \tag{2}$$

$$[M_{ab}, M_{cd}] = i\Big(\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}\Big)\,, \tag{3}$$

- P_a are translation generators and
- $M_{ab} = L_{ab} + S_{ab}$ are four dimensional rotations generators
  They consist of orbital part $L_{ab} = x_a P_b - x_b P_a$ and spin part S_{ab}.

Poincare Casimir operators

- Casimir operators are expressions which commute with all group generators and allow us to label the irreducible representations
- For $P^2 > 0$ there are two Casimir operators which eigenvalues are mass m and spin s

$$P^2 = m^2 \,, \qquad W^2 = -m^2 s(s+1) \,. \tag{4}$$

- Pauli-Lubanski vector

$$W_a = \frac{1}{2} \varepsilon_{abcd} M^{bc} P^d \,. \tag{5}$$

Principle field equations

- We will introduce field $\Psi^A(x)$ where A contains the set of vector and spinor indices. In order to separate states which describe definite particles we should impose some constraints on the field $\Psi^A(x)$
- Since in field theory particles are defined by mass and spin it is natural to use just operators whose eigenvalues are mass and spin
- We will postulate principle field equations for arbitrary spin as representation of relations (4)

$$(P^2)^A{}_B \Psi^B(x) = m\Psi^A(x) \,, \qquad \mathcal{S}^A{}_B \Psi^B(x) = s(s+1)\Psi^A(x) \,. \tag{6}$$

  These equations are Poincare covariant because Casimir operators commute with all Poincare generators
- In particular, Casimir operators commute mutually. Since commuting observables have a complete set of common eigenfunctions we are able to impose both Casimir operators to the same field $\Psi^A(x)$.

Representation of Casimir operators

- To find representation of Casimir operators we need representation of Poincare algebra generators, momentum P_a and spin S_{ab}
- Representation of momentum P_a is well known from quantum mechanics $(P_a)^A{}_B \to i\delta^A_B\partial_a$ and it is spins independent.
- Spin generators $(S_{ab})^A{}_B$ act as derivatives

$$(S_{ab})^{AB}{}_{CD} = (S_{ab})^A{}_C\delta^B_D + \delta^A_C(S_{ab})^B{}_D \tag{7}$$

The initial expression for Dirac spinor and vector fields are

$$(S_{ab})^\alpha{}_\beta = \frac{i}{4}[\gamma_a, \gamma_b]^\alpha{}_\beta\,, \qquad (S_{ab})^c{}_d = i\left(\delta^c_a\eta_{bd} - \delta^c_b\eta_{ad}\right) \tag{8}$$

and we can find representations for all other fields using recurrence relation

Standard momentum

- The next step is to find projectors on irreducible representations
  In order to to achieve that it is useful to go to standard momentum.
- For massive case where $p^2 = m^2$ we can chose rest frame momentum $k^a = (m, 0, 0, 0)$ as standard momentum.
- With this choice the Klein-Gordon equation is solved.
- Later we can express any momentum p^a as Lorentz transformation of k^a

$$p^a = L^a{}_b(p) k^b \,. \tag{9}$$

Principle field equations for standard momentum

- For standard momentum differential equation become algebraic ones. After solving algebraic equation we can go back to p^a dependent solutions and then to solution in coordinate representation.
- The spin equation for standard momentum takes a form

$$\mathcal{S}^A{}_B \Psi^B(k) = s(s+1)\Psi^A(k)\,, \qquad (10)$$

where

$$\mathcal{S}^A{}_B = (S_i^2)^A{}_B\,, \qquad (S_i)^A{}_B = \frac{1}{2}\,\varepsilon_{ijk}(S_{jk})^A{}_B\,. \qquad (11)$$

- Note that instead of 6 components of spin operator S_{ab} in the frame of standard momentum we have left with 3 components S_i, which are generators of space rotations. They form a subgroup known as little group for massive Poinacare case.

Projection operators and equation of motion

- To have nontrivial solution for function Ψ^A the characteristic polynomial must vanish

$$\det\left(\mathcal{S}^A{}_B - \lambda\delta^A_B\right) = 0\,, \qquad \lambda \equiv s(s+1)\,. \tag{12}$$

The values s_i, corresponding to the eigenvalues λ_i, are spins of irreducible representations.

- The representations of eigenfunctions Ψ_i^A with definite spin have a form

$$\Psi_i^A = (P_i)^A{}_B\Psi^B\,, \qquad (P_i)^A{}_B = \frac{\left[\prod_{j\neq i}^n\left(\mathcal{S}-\lambda_j\right)\right]^A{}_B}{\prod_{j\neq i}^n\left(\lambda_i-\lambda_j\right)}\,, \qquad i=\{1,2,\cdots,n\}\,, \tag{13}$$

where $(P_i)^A{}_B$ are corresponding projection operators

Scalar field

- We are going to confirm above massive principle field equations for particular spins:
  - Klein-Gordon equation for scalar fields ($s = 0$)
  - Dirac equation for spinors ($s = \frac{1}{2}$)
  - equation for massive vector fields ($s_0 = 0$ and $s_1 = 1$).
  - Rarita-Schwinger equation ($s_0 = \frac{1}{2}$ and $s_1 = \frac{3}{2}$)
- A scalar field has no indices $\Psi^A(x) \to \varphi(x)$, so that $(S_{ab})^A{}_B \to 0$. Equation (12) produces $\lambda = 0$ and consequently $s = 0$. We are left with the Klein-Gordon equation

$$\left(\partial^2 + m^2\right)\varphi(x) = 0\,. \tag{14}$$

Dirac field

- For Dirac field $\Psi^A(x) \to \psi^\alpha(x)$ and representation of spin operator is

$$(S_{ab})^A{}_B \to (S_{ab})^\alpha{}_\beta = \frac{i}{4}[\gamma_a, \gamma_b]^\alpha{}_\beta \,,$$

$$S_i = \frac{1}{2}\,\varepsilon_{ijk} S_{jk} = \frac{i}{4}\varepsilon_{ijk}\gamma_j\gamma_k \,, \qquad \mathcal{S}^\alpha{}_\beta = [(S_i)^2]^\alpha{}_\beta = \frac{3}{4}\delta^\alpha_\beta \,. \tag{15}$$

- Then the spin equation takes the form

$$\mathcal{S}^\alpha{}_\beta \psi^\beta(k) = \lambda\psi^\alpha(k) \,, \qquad \lambda \equiv s(s+1) \tag{16}$$

and since $\mathcal{S}^\alpha{}_\beta$ is diagonal we obtain

$$\det(\mathcal{S} - \lambda)^\alpha{}_\beta = \left(\lambda - \frac{3}{4}\right)^4 = 0 \,. \tag{17}$$

- Therefore, $\lambda = \frac{3}{4}$ which produces spin $s = \frac{1}{2}$ There is only one trivial projection operator $P^\alpha{}_\beta = \delta^\alpha_\beta$. We obtain Klein-Gordon equation for all components $(\partial^2 + m^2)\psi^\alpha(x) = 0$.
- We can linearize it in the form of Dirac equation

$$(i\gamma^a\partial_a + m)\psi^\alpha(x) = 0 \,, \tag{18}$$

where γ^a are constant matrices. In fact, Dirac equation produces Klein-Gordon equation if

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \,. \tag{19}$$

Vector field 1

- For vector field we have $\Psi^A \to V^a$, and from representation of spin operators we obtain

$$(S_i)^a{}_b = \frac{1}{2}\,\varepsilon_{ijk}(S_{jk})^a{}_b = i\,\varepsilon_{ijk}\delta^a_j\,\eta_{kb}\,, \qquad \mathcal{S}^a{}_b = (S_i^2)^a{}_b = -2\delta_{jk}\delta^a_j\,\eta_{kb}\,. \tag{20}$$

Therefore, the spin equation in the rest frame takes the form

$$\mathcal{S}^a{}_b V^b(k) = \lambda V^a(k)\,. \qquad \lambda \equiv s(s+1) \tag{21}$$

- The consistency condition produces

$$\det(\mathcal{S}-\lambda)^a{}_b = -\lambda(2-\lambda)^3 = 0\,, \tag{22}$$

with solutions for eigenvalues $\lambda_0 = 0\,,\ \lambda_1 = 2$ and for spins $s_0 = 0\,,\ s_1 = 1$.
- We obtain two projectors

$$(P_0)^a{}_b(k) = \delta^a_b - \frac{\mathcal{S}^a{}_b}{2} = \delta^a_0\delta^0_b\,, \qquad (P_1)^a{}_b(k) = \frac{\mathcal{S}^a{}_b}{2} = \delta^a_b - \delta^a_0\delta^0_b\,. \tag{23}$$

- To find irreducible representations in arbitrary frame we should boost corresponding equation of the rest frame. Then for projection operators we obtain standard form of longitudinal and transversal projection operators

$$(P^L)^a{}_b(p) = \frac{p^a p_b}{p^2}\,, \qquad (P^T)^a{}_b(p) = \delta^a_b - \frac{p^a p_b}{p^2}\,. \tag{24}$$

Vector field 2

- Therefore, solution of the spin equation produces two irreducible representations in coordinate space

$$s=0: \quad V_L^a(x)=\frac{\partial^a\partial_b}{\partial^2}V^b(x)\,, \qquad s=1: \quad V_T^a(x)=V^a(x)-V_L^a(x)\,. \tag{25}$$

- Equations of motion for massive vector fields, for spins 0 and 1 have a form

$$\left(\partial^2+m^2\right)V_L^a(x)=0\,, \qquad \left(\partial^2+m^2\right)V_T^a(x)=0\,. \tag{26}$$

If we introduce notation $\varphi=\partial_b V^b$ so that

$$V_a^L=\frac{\partial_a}{\partial^2}\varphi\,, \qquad V_a^T=U_a\,, \tag{27}$$

we can rewrite above equations as

$$\left(\partial^2+m^2\right)\varphi(x)=0\,, \qquad \left(\partial^2+m^2\right)U_a(x)=0\,, \tag{28}$$

where U_a satisfy condition $\partial_a U^a=0$. This condition reduces four components of the field U_a to three degrees of freedom. It provides positivity of vector field energy.

Massive Rarita-Schwinger field 1 (Spin operator)

- For Rarita-Schwinger fields we have $A, B \to (a\alpha), (b\beta)$ and $\Psi^A \to \psi^{a\alpha}$ where a, b are vector and α, β spinor indices, so that spin equation takes the form

$$\mathcal{S}^{a\alpha}{}_{b\beta}\psi^{b\beta} = \lambda\psi^{a\alpha}\,, \qquad \mathcal{S}^{a\alpha}{}_{b\beta} = (S_i^2)^{a\alpha}{}_{b\beta}\,, \qquad \lambda = s(s+1)\,. \tag{29}$$

Spin operator for Rarita- Schwinger field is

$$(S_{ab})^{c\alpha}{}_{d\beta} = (s_{ab})^c{}_d\delta^\alpha_\beta + \delta^c_d(f_{ab})^\alpha{}_\beta\,, \tag{30}$$

can be express in terms of spin operators for Dirac fields $(f_{ab})^\alpha{}_\beta$ and spin operators for vector fields $(s_{ab})^c{}_d$. Then we have

$$(S_i)^{c\alpha}{}_{d\beta} = (s_i)^c{}_d\delta^\alpha_\beta + \delta^c_d(f_i)^\alpha{}_\beta\,, \tag{31}$$

where for vector and Dirac fields

$$(s_i)^c{}_d = i\,\varepsilon_{ijk}\delta^c_j\,\eta_{kd}\,, \qquad (f_i)^\alpha{}_\beta = \frac{i}{4}\,\varepsilon_{ijk}(\gamma_j\gamma_k)^\alpha{}_\beta\,. \tag{32}$$

- Consequently, after some calculation we obtain Rarita-Schwinger spin operator

$$\mathcal{S}^{a\alpha}{}_{b\beta} = 3\Big(\frac{1}{4}\delta^a_b - \delta^a_i\,\eta_{ib}\Big)\delta^\alpha_\beta - \delta^a_i\,\eta_{jb}(\gamma_i\gamma_j)^\alpha{}_\beta\,. \tag{33}$$

Massive Rarita-Schwinger field 2 (Projectors)

- We can rewrite Rarita-Schwinger spin operator in the form

$$\mathcal{S}^{a\alpha}{}_{b\beta} = \frac{3}{4}(\pi_0)^a{}_b\delta^\alpha_\beta + \frac{15}{4}(\pi_1)^a{}_b\delta^\alpha_\beta - 3(\pi)^{a\alpha}{}_{b\beta}\,, \tag{34}$$

where projectors of vector case $(\pi_0)^a{}_b$ and $(\pi_1)^a{}_b$ has been defined in (23) and

$$(\pi)^{a\alpha}{}_{b\beta} = \frac{1}{3}\delta^a_i\,\eta_{jb}(\gamma_i\gamma_j)^\alpha{}_\beta\,. \tag{35}$$

- With the help of expression $\delta^a_b = (\pi_0)^a{}_b + (\pi_1)^a{}_b$ we obtain

$$\mathcal{S}^{a\alpha}{}_{b\beta} - \lambda\delta^a_b\delta^\alpha_\beta = \left(\frac{3}{4} - \lambda\right)(P_0)^{a\alpha}{}_{b\beta} + \left(\frac{15}{4} - \lambda\right)(P_1)^{a\alpha}{}_{b\beta}\,, \tag{36}$$

where we introduced expressions

$$(P_0)^{a\alpha}{}_{b\beta} = (\pi_0\delta + \pi)^{a\alpha}{}_{b\beta}\,, \quad (P_1)^{a\alpha}{}_{b\beta} = (\pi_1\delta - \pi)^{a\alpha}{}_{b\beta}\,. \tag{37}$$

- It is easy to check that P_0 and P_1 are projectors

$$P_0^2 = P_0\,, \quad P_1^2 = P_1\,, \quad P_0P_1 = 0\,, \quad P_0 + P_1 = 1\,. \tag{38}$$

Massive Rarita-Schwinger field 3 (Spins)

- The consistency condition produces

$$\det(\mathcal{S}-\lambda)^{a\alpha}{}_{b\beta} = \left(\frac{3}{4}-\lambda\right)^{d_0}\left(\frac{15}{4}-\lambda\right)^{d_1} \det P_0 \; \det P_1 = 0\,, \tag{39}$$

  The solutions for eigenvalues are $\lambda_0 = \frac{3}{4}$, $\lambda_1 = \frac{15}{4}$ and for spins $s_0 = \frac{1}{2}$, $s_1 = \frac{3}{2}$.

- Finally, Rarita-Schwinger projectors in arbitrary frame are

$$\begin{aligned}(P_0)^{a\alpha}{}_{b\beta}(p) &= (\pi_0)^a{}_b\delta^\alpha_\beta + \frac{1}{3}(\pi_1)^a{}_c(\pi_1)_b{}^d(\gamma^c\gamma_d)^\alpha{}_\beta\,,\\ (P_1)^{a\alpha}{}_{b\beta}(p) &= (\pi_1)^a{}_b\delta^\alpha_\beta - \frac{1}{3}(\pi_1)^a{}_c(\pi_1)_b{}^d(\gamma^c\gamma_d)^\alpha{}_\beta\,.\end{aligned} \tag{40}$$

Massive Rarita-Schwinger field 4 (Spin-$\frac{3}{2}$)

- Spin-$\frac{3}{2}$ Rarita-Schwinger equations in arbitrary frame consists of Dirac equation for the vector-spinor field plus supplementary condition.
  They can be combined into one equation as linear combination of Dirac equation and supplementary conditions

$$\Big[(i\hat{\partial} - m)P_1 + A\pi_0 + B\pi + C\pi_{*0} + D\pi_{0*}\Big]^a{}_b \psi^{b\alpha}(x) = 0\,. \tag{41}$$

- This equation contain singularities ∂^{-2}. So, we are going to chose coefficients A, B, C and D in such a way that this equation becomes regular.

Massive Rarita-Schwinger field 5 (original Rarita-Schwinger equation)

- Finaly we obtain exactly equation from Nieuwenhuizen article

$$\Big[i\hat{\partial}(P_1 - 2\pi) - m\Big(P_1 - 2\pi - \sqrt{3}\Big(\pi_{*0} + \pi_{0*}\Big)\Big)\Big]^a{}_b \psi^b = 0\,. \tag{42}$$

- We can rewrite it in the form

$$\varepsilon^{abcd}\gamma^5\gamma_b\partial_c\psi_d(x) + \frac{m}{2}[\gamma^a, \gamma_b]\psi^b(x) = 0\,. \tag{43}$$

This is original Rarita-Schwinger equation which can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}_a\Big(\varepsilon^{abcd}\gamma^5\gamma_b\partial_c - im\sigma^{ad}\Big)\psi_d\,, \qquad \sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]\,. \tag{44}$$

Casimir operators and standard momentum for massless Poincare group

- In massless case, square of momentum vanishes $P^2 = 0$. The Casimir operators for massless Poincare group, is helicity λ. The covariant definition of helicity is

$$W_a = \lambda P_a \,, \qquad W_a = \frac{1}{2} \varepsilon_{abcd} S^{bc} P^d \,, \tag{45}$$

  where W_a is Pauli-Lubanski vector and S_{ab} is spin parts of Lorentz generators. Since $P^a W_a = 0$ we can conclude from (45) that $P^2 = 0$, as it should be for massless case.
- It is possible to use non covariant definition of helicity $\lambda = S_i n_i$ as projection of space part of spin generator $S_i = \frac{1}{2} \varepsilon_{ijk} S_{jk}$ to momentum axis $n_i = \frac{p_i}{|p_i|^2}$. It produces the same spectrum for λ but in that case we are losing the crucial possibility to insist on Lorentz invariance which will product gauge transformations.
- In massless case, when Lorentz invariant function of momentum vanish, $p^2 = 0$, we can chose standard momentum as $k^a = (1, 0, 0, -1)$.

Principle field equations for massless fields

- We can postulate principle field equations for massless fields $\Psi^A(x)$ with helicity λ

$$(W_a)^A{}_B \Psi^B(x) = \lambda (P_a)^A{}_B \Psi^B(x) . \tag{46}$$

We claim that this equations contains all massless free field equations with arbitrary helicity, known in classical field theory.

- Principle field equations for standard momentum are

$$\begin{gathered} (S_{12})^A{}_B \Psi^B(k) = \lambda \Psi^A(k) , \\ (\Pi_1)^A{}_B \Psi^B(k) = 0 , \qquad (\Pi_2)^A{}_B \Psi^B(k) = 0 . \end{gathered} \tag{47}$$

- where

$$\begin{gathered} W_0 = S_{12} = W_3 , \\ W_1 \equiv \Pi_2 = S_{02} - S_{32} , \qquad -W_2 \equiv \Pi_1 = S_{01} - S_{31} . \end{gathered} \tag{48}$$

All three generators annihilate standard momentum. So group element $W^a{}_b$, leaves k^a invariant $W^a{}_b k^b = k^a$, which is definition of little group in massless Poincare case.

Spectrum of helicities

- We are going to solve eigenproblem of operator $(S_{12})^A{}_B$ which will produces spectrum of helicities λ_i $(i = 1, 2, \cdots, n)$. In order that first equation (47) has nontrivial solutions its characteristic polynomial must vanish

$$\det\left((S_{12})^A{}_B - \lambda\sigma\delta^A_B\right) = 0\,. \tag{49}$$

The zeros of characteristic polynomial are eigenvalues λ_i.

- Next we can construct projection operators $(P_i)^A{}_B$ corresponding to helicities λ_i

$$(P_i)^A{}_B = \frac{\left[\prod_{j\neq i}^n \left(S_{12} - \lambda_j\delta\right)\right]^A{}_B}{\prod_{j\neq i}^n \left(\lambda_i - \lambda_j\right)}\,, \tag{50}$$

and obtain corresponding eigenfunctions

$$\Psi_i^A(k) = (P_i)^A{}_B \Psi^B(k)\,. \tag{51}$$

Lorentz transformations induce gauge transformations

- We solved first equation (47) and it produces complete set of eigenvalues and eigenfunctions. But we have additional two conditions. It is of great importance to investigate the compatibility of all equations (47).
- In equations for standard momentum (47) operators Π_1 and Π_2 should annihilate field $\Psi^A(k)$. But explicit calculation shows that this does not happen in some physically relevant cases. Since equations (47) are condition for Lorentz invariance, violation of these conditions breaks Lorentz invariance. To measure this violation we introduce expression

$$\delta\Psi_i^A(\varepsilon_1, \varepsilon_2)(k) = i\Big(\varepsilon_1\Pi_1 + \varepsilon_2\Pi_2\Big)^A{}_B \Psi_i^B(k)\,, \tag{52}$$

  where ε_1 and ε_2 are some parameters.
- Fields which violate Lorentz invariance are gauge dependent fields and we will simply call them gauge fields. They are not representation of Poincare group.

Vector field and Maxwell equations

- The case of massless vector field is most important. Besides representing electromagnetic interaction we will see that solutions for massless tensor fields of arbitrary rank comes down to the case of vector field.
- For vector field we have $A, B \to a, b$, $\Psi^A \to V^a$ and

$$(S_{ab})^A{}_B \to \Big(S_{ab}\Big)^c{}_d = i\Big(\delta^c_a\,\eta_{bd} - \delta^c_b\,\eta_{ad}\Big)\,, \qquad \Rightarrow \qquad (S_{12})^a{}_b = i\Big(\delta^a_1\eta_{2b} - \delta^a_2\eta_{1b}\Big)\,. \tag{53}$$

- The consistency condition requires that characteristic polynomial vanishes. It means

$$\det\Big(\mathcal{S}_{12} - \lambda\sigma\Big)^a{}_b = \lambda^2(\lambda-1)(\lambda+1) = 0\,, \tag{54}$$

with solutions for helicities

$$\lambda_0 = 0\,, \qquad \lambda_1 = 1\,, \qquad \lambda_{-1} = -1\,. \tag{55}$$

Basic vectors

- We introduce basic vectors

$$k^a = \delta_0^a - \sigma\delta_3^a\,, \quad q^a = \delta_0^a + \sigma\delta_3^a\,, \quad \breve{p}^a_\pm = \delta_1^a \pm i\sigma\delta_2^a\,. \tag{56}$$

- Projection operators in terms of basic vectors take the form

$$(P_{0+})^a{}_b(k) = \frac{1}{2}q^a k_b\,, \qquad (P_{0-})^a{}_b(k) = \frac{1}{2}k^a q_b\,, \qquad (P_{\pm1})^a{}_b(k) = -\frac{1}{2}\breve{p}^a_\pm(\breve{p}_\mp)_b\,. \tag{57}$$

- The eigenfunctions of operator $(S_{12})^a{}_b$ are

$$V_i^a(k) = (P_i)^a{}_b V^b\,. \tag{58}$$

- It is easy to check that for $e_i^a = \{k^a, q^a, \breve{p}^a_+, \breve{p}^a_-\}$

$$(S_{12})^a{}_b e_i^b = \lambda_i e_i^a\,, \tag{59}$$

Consequently, basic vector e_i^a carries helicity λ_i. In particular, basic vectors $\breve{p}^a_\pm$ carry helicities ±1 and both basic vectors k^a and q^a carry helicities 0. If $n_\pm$ are numbers of vectors $\breve{p}^a_\pm$ then its helicity is

$$\lambda = n_+ + n_-\,. \tag{60}$$

Gauge transformations of massless vector fields

- Gauge transformations of vector components are

$$\delta V_i^a(\varepsilon_1, \varepsilon_2)(k) = i\Big(\varepsilon_1 \Pi_1 + \varepsilon_2 \Pi_2\Big)^a{}_b V_i^b(k)\,, \tag{61}$$

- Gauge transformations of basic vectors have simple form

$$\delta k^a = 0\,, \qquad \delta q^a = \varepsilon_+ \breve{p}_-^a + \varepsilon_- \breve{p}_+^a\,, \qquad \delta \breve{p}_\pm^a = \varepsilon_\pm k^a\,. \qquad (\varepsilon_\pm = \varepsilon_1 \pm i\sigma\varepsilon_2) \tag{62}$$

- Only component $V_{0-}^a(k)$ is gauge invariant, $\delta V_{0-}^b(k) = 0$, and consequently only this part is irreducible representation of Poincae group
- The other components $V_{0+}^a(k)$ and $V_{\pm 1}^b(k)$ are not gauge invariant and they transform as

$$\delta V_{0+}^a(k) = \omega_+ \breve{p}_-^a + \omega_- \breve{p}_+^a\,, \qquad \omega_\pm(k) = \frac{1}{2} k_a V^a \varepsilon_\pm\,,$$

$$\delta V_{\pm 1}^a(k) = k^a \Omega_\pm\,, \qquad \Omega_\pm(k) = -\frac{1}{2} (\breve{p}_\mp)_b V^b \varepsilon_\pm\,. \tag{63}$$

Action for massless vector field with helicities $\lambda = \pm 1$ produces Maxwell equations

- The electromagnetic interaction is symmetric under space inversion. Therefore, instead of two components $V^a_{+1}(k)$ and $V^a_{-1}(k)$ we will introduce one vector field $A^a(k)$

$$A^a(k) = \alpha V^a_{+1}(k) + \beta V^a_{-1}(k) . \tag{64}$$

  We also introduce one parameter $\Omega(k) = \alpha\Omega_+(k) + \beta\Omega_-(k)$ instead of two components $\Omega_+(k)$ and $\Omega_-(k)$ related by space inversion.
- Gauge transformation in coordinate representation is

$$\delta A^a(x) = \partial^a \Omega(x) . \tag{65}$$

- Action should depend only on the gauge invariant combination of vector fields, in fact of field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. Since it must be scalar we can take

$$I_0 = -\frac{1}{4} \int d^4x F_{ab} F^{ab} . \tag{66}$$

Interaction with other fields

- Interaction with other fields can be described by the action

$$I_{int}\left(A^a\right) = \int d^4x A^a J_a \,. \tag{67}$$

- Requirement for gauge invariace $I_{int}\left(A^a + \delta A^a\right) = I_{int}\left(A^a\right)$ after partial integration produces $\int d^4x \Omega\, \partial^a J_a = 0$. For arbitrary Ω we obtain conservation condition on current

$$\partial^a J_a = 0 \,. \tag{68}$$

- The complete action is

$$I = I_0 + I_{int} = \int d^4x \left(-\frac{1}{4} F_{ab} F^{ab} + A^a J_a \right) . \tag{69}$$

Variation with respect to A^a produces Maxwell equations.

Second rank tensor

- In the case of second rank tensor we have $A \to (ab)$ and $\Psi^A \to T^{ab}$.
- Representation of spin operator for second ranktensor takes a form

$$(S_{12})^{ab}{}_{cd} = \rho^a{}_c \delta^b_d + \delta^a_c \rho^b{}_d = \left(\rho\delta + \delta\rho\right)^{ab}{}_{cd} \,, \tag{70}$$

$$(P_0)^a{}_b = (\pi_0)^a{}_b \,, \qquad (P_{\pm 1})^a{}_b = (\pi_\pm)^a{}_b \,. \qquad (S_{12})^a{}_b = \rho^a{}_b \,, \tag{71}$$

Projection operators

- Using relations $\rho = \pi_+ - \pi_-$ and $\delta = \pi_0 + \pi_+ + \pi_-$ we obtain

$$S_{12} = P_1 + 2P_2 - P_{-1} - 2P_{-2}\,, \qquad \delta\delta = P_1 + P_2 + P_{-1} + P_{-2} + P_0\,, \tag{72}$$

where

$$P_0 = \pi_0\pi_0 + \pi_+\pi_- + \pi_-\pi_+\,, \qquad P_{\pm 1} = \pi_0\pi_\pm + \pi_\pm\pi_0\,, \qquad P_{\pm 2} = \pi_\pm\pi_\pm\,. \tag{73}$$

Using the fact that π_0, π_+ and π_- are projectors we can conclude that $(P_i)^{ab}{}_{cd}$ $(i = 0, \pm 1, \pm 2)$ are projectors also.

Spectrum of second rank tensor

- For second rank tensor characteristic matrix takes the form

$$S_{12} - \lambda\sigma\delta\delta = -\lambda\sigma P_0 + (1-\lambda\sigma)P_1 + (2-\lambda\sigma)P_2 - (1+\lambda\sigma)P_{-1} - (2+\lambda\sigma)P_{-2}\,. \qquad (74)$$

- Since $(P_i)^{ab}{}_{cd}$ are projectors, characteristic polynomial is already factored and we have

$$\det\Big(S_{12} - \lambda\sigma\delta\delta\Big) = \Big(-\lambda\Big)^6\Big(1-\lambda^2\Big)^4\Big(4-\lambda^2\Big)\prod_{i=-2}^{2}\det P_i = 0\,. \qquad (75)$$

- Using the fact that $\det P_i \neq 0$ we can find that spectrum of helicities is

$$\lambda_0 = 0\,, \qquad \lambda_{\pm 1} = \pm 1\,, \qquad \lambda_{\pm 2} = \pm 2\,. \qquad (76)$$

- Consequently, the eigenfunctions of operator S_{12} in full notation have the form

$$T_i^{ab}(k) = (P_i)^{ab}{}_{cd}T^{cd}(k)\,, \qquad (i = 0, \pm 1, \pm 2) \qquad (77)$$

- Second rank tensor with helicities $\lambda = \pm 2$ is of particular importance since its symmetric part (metric) contribute to general relativity.

Second rank tensor with helicities $\lambda = \pm 2$ and general relativity in weak field approximation

- Second rank tensor with highest helicities $\lambda = \pm 2$ is symmetric tensor

$$T^{ab}_{\pm 2}(k) = (P_{\pm 2})^{ab}{}_{cd} T^{cd}(k) = \frac{1}{4}(\breve{p}_{\mp})_c(\breve{p}_{\mp})_d T^{cd}(k)\breve{p}^a_{\pm}\breve{p}^b_{\pm} \,, \tag{78}$$

Since gravitation interaction is symmetric under space inversion it is common to treat both components as a single particle called graviton. So, instead of two polarizations $T^{ab}_{+2}(k)$ and $T^{ab}_{-2}(k)$, we will introduce one symmetric tensor

$$h^{ab}(k) = \alpha T^{ab}_{+2}(k) + \beta T^{ab}_{-2}(k) \,. \tag{79}$$

We also introduce one parameter $\Omega^a(k) = \alpha\Omega^a_+(k) + \beta\Omega^a_-(k)$ instead of two components $\Omega^a_+(k)$ and $\Omega^a_-(k)$ related by space inversion.

Action

- Gauge transformation in coordinate representation has a form

$$\delta h^{ab}(x) = \partial^a \Omega^b(x) + \partial^b \Omega^a(x)\,. \tag{80}$$

- Equation of motion with respect to h^{ab} has two indices. We are going later to include interaction and so we will take Einstein tensor for free field equation

$$G^{ab}(x) = R^{ab} - \frac{1}{2}\eta^{ab} R = \partial_c \partial^b h^{ac} - \partial^2 h^{ab} - \partial^a \partial^b h + \partial^a \partial_c h^{cb} - \eta^{ab}\Big(\partial_c \partial_d h^{cd} - \partial^2 h\Big) = 0\,.$$

- Field equation (81) can be obtained from the action

$$I_0(h^{ab}) = \int d^4x \Big(\frac{1}{2}\partial_c h_{ab} \partial^c h^{ab} - \partial_c h_{ab} \partial^b h^{ac} + \partial_c h^{ac} \partial_a h - \frac{1}{2}\partial_a h \partial^a h\Big)\,. \tag{82}$$

Interaction with matter

- Interaction with matter field Φ^A we can describe with the action quadratic in h^{ab}

$$I_{int}(h^{ab}) = \int d^4x \mathcal{L}(h^{ab}, \Phi^A)\,. \tag{83}$$

Note that the integration measure is the flat Minkowski measure d^4x, as Lagrangian is already quadratic in h^{ab}.

- From requirement that $I_{int}(h^{ab})$ is gauge invariant $I_{int}(h^{ab} + \delta h^{ab}) = I_{int}(h^{ab})$ we have

$$\int d^4x \delta h^{ab} \frac{\partial \mathcal{L}}{\partial h^{ab}} = \int d^4x \Big(\partial^a \Omega^b + \partial^b \Omega^a\Big) \Theta_{ab} = 0\,. \qquad \Theta_{ab} = \frac{\partial \mathcal{L}}{\partial h^{ab}} \tag{84}$$

Using partial integration and fact that Θ_{ab} is symmetric tensor we obtain that Θ_{ab} is conserved energy-momentum tensor, $\partial_a \Theta^{ab} = 0$.

- The complete action is $I(h^{ab}) = I_0(h^{ab}) + I_{int}(h^{ab})$. Its variation with respect to h^{ab} produces complete Einstein equations in weak field approximation

$$G_{ab}(h^{ab}) = \Theta_{ab}\,. \tag{85}$$

Second rank tensor with helicities $\lambda = \pm 1$ produces Maxwell equations

- Beside highes helicities $\lambda = \pm 2$, second rank tensor has also components with helicities $\lambda = \pm 1$. These components describe Maxwell equations.
- For symmetric tensor we obtain

$$(T^S_{\pm 1(-)})^{(ab)}(k) = \mathbf{T}^S_{\mp 2}\left(k^a \breve{p}^b_\pm + k^b \breve{p}^a_\pm\right) = k^a \varphi^b_\pm(k) + k^b \varphi^a_\pm(k)\,, \tag{86}$$

and for antisymmetric one

$$(T^A_{\pm 1(-)})^{[ab]}(k) = \mathbf{T}^A_{\mp 2}\left(k^a \breve{p}^b_\pm - k^b \breve{p}^a_\pm\right) = k^a A^b_\pm(k) - k^b A^a_\pm(k)\,, \tag{87}$$

Here $\varphi^a_\pm = \mathbf{T}^S_{\mp 2}\breve{p}^a_\pm$ and $A^a_\pm = \mathbf{T}^A_{\mp 2}\breve{p}^a_\pm$ are new fields defined above. Note that both $\varphi^a_\pm$ and $A^a_\pm$ are proportional to $\breve{p}^a_\pm$, as well as fields $V^a_\pm$ in
- The field $\varphi^a(x)$ behaves like electromagnetic field. It is vector field, it has the same helicity, the same gauge transformation and the same number of degrees of freedom.
- Antisymmetric part is gauge invariant and it corresponds to field strength.

Massless tensors with arbitrary rank

- We are going to solve most general equation for integer helicity

$$(S^A{}_B - \lambda\delta^A_B)\Psi^B = 0\,, \tag{88}$$

where Ψ^A is tensor with rank n so that $\Psi^A = T^{a_1 a_2 \cdots a_n}$. The problem comes down to that of vector fields.

- Using $\rho = \pi_+ - \pi_-$ and $\delta = \pi_0 + \pi_+ + \pi_-$ we obtain

$$(S_{12})^A{}_B = \sum_{k=1}^{n} k\Big(P_k - P_{-k}\Big)^A{}_B\,, \tag{89}$$

and delta function for tensors with arbitrary rank

$$\delta^A_B = \delta^n = (\pi_0 + \pi_+ + \pi_-)^n = \sum_{k=1}^{n}\Big(P_k + P_{-k}\Big) + P_0\,. \tag{90}$$

- Here $(P_k)^A{}_B$ is sum of all terms $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_n}$ such that $\sum_{m=1}^{n} i_m = k$ where we count π_+ as π_1 and π_- as π_{-1}. Note that as all expressions P_k are projectors $P_k P_q = \delta_{kq} P_k$ where $(k, q = 0, \pm1, \pm2, \cdots, \pm n)$. The multiplication factor k in (89) is consequence of combinatorics.

Spectrum for arbitrary rank tensors

- Since $(P_k)^A{}_B$ are projection operators $S_{12} - \lambda$ is diagonalized and we have

$$(S_{12})^A{}_B - \lambda\delta^A_B = -\lambda(P_0)^A{}_B + \sum_{k=1}^{n}\Big[(k-\lambda)(P_k)^A{}_B - (k+\lambda)(P_{-k})^A{}_B\Big]\,. \qquad (91)$$

- Helicities for n rank tensor are

$$\lambda_0 = 0\,, \qquad \lambda_k = k\,, \qquad \lambda_{-k} = -k\,, \qquad k = (1, 2, \cdots, n)\,. \qquad (92)$$

- The eigenfunctions, which are our candidates for irreducible representations, for massless tensors with arbitrary rank are

$$\Psi^A_m = (P_m)^A{}_B \Psi^B\,. \qquad (m = 0, \pm1, \pm2, \cdots, \pm n) \qquad (93)$$

Fronsdal's action for highest helicity

- Gauge transformation

$$\delta h_A(k) = \sum_{i=1}^{n} \partial_{a_i} \Omega_{A_i}(k) \,. \tag{94}$$

- Fronsdal's action for highest helicity

$$S_n(h) = \frac{1}{2} \int d^4x \Big[-\partial^a h^A \partial_a h_A + n \partial_a h^{aA_i} \partial^b h_{bA_i} + n(n-1) \partial_a \partial_b h^{abA_{ij}} h^c{}_{cA_{ij}} + \frac{n(n-1)}{2} \partial^c h_b{}^{bA_{ij}} \partial_c h^a{}_{aA_{ij}} + \frac{n(n-1)(n-2)}{4} \partial_c h_b{}^{bcA_{ijk}} \partial^b h^a{}_{abA_{ijk}} \Big] \,. \tag{95}$$

- For $n = 1$ we obtain Maxwell action and for $n = 2$ Einstein action in weak field approximation.