Geometry & Physics

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The root of the present GPF group is the Theoretical physics project (of at-the-time Ministry of Science) that existed 50 years ago and had two parts: Quantum Mechanics and QFT/Gravity. The leading researcher of the QM subproject was Zvonko Marić while the QFT/GR part was led by Đorđe Živanović.
(This is the large-scale framework as I remember it: the accurate details probably differ.)

At the time when I fully joined, the group was led by the young and enthusiastic trio Milutin Blagojević - Đorđe Šijački - Dragan Popović, who was the youngest. My main (formal) relation to Dragan was that for several years I did tutorials for the course Particle physics which he had taught. The group had regular seminars on Fridays (at 10 a.m.!), where a lot was learned and discussed, and workshops like the Danube workshop in the photo (on which your PhD advisors are just babies).

The first and most important unification of geometry and physics was Einstein's theory of General Relativity.

It created the yet-lasting paradigm in theoretical physics that fundamental physics should be expressed as geometry (or through symmetries?)

Some of the consequences of Einstein's bold assumptions about the nature of spacetime are still, 100 years later, being experimentally verified, like gravitational waves (2015/2016) and black holes (2012/2022).

But the big question still is, how to push gravity / geometry / symmetry relation forward, to quantum physics?

Einstein used his intuition to recognise the main physical principle(s), and then applied mathematics (however unusual at the time) to extend standard classical nonrelativistic physics.

Today too, we are trying to implement our, gained in the meanwhile, intuition about the quantum world, and/or develop mathematics, in order to describe spacetime and gravity on very small scales. Unfortunately, at the moment, our intuition about quantum gravity is not based on the experimental facts.

The basic principle used in GR is the equivalence principle; it can be formulated in several similar ways. According to MB, one can define its following versions.

EP: a non-inertial reference system cannot be locally experimentally distinguished from the gravitational field

EP': the physics laws in s freely falling reference frame are the same as in special relativity
(or, Poincaré group is the local group of symmetry)

shift to non-inertial reference systems:
GRP: the form of the laws of physics is the same in all reference frames

GCP: the form of the laws of physics is independent of the choice of coordinates

Equivalence principle is mostly about symmetries, but what it states as well is the classical limit. What is the corresponding mathematics?

Physical fields like metric, curvature etc. are tensors with respect to coordinate transformations, e.g. $g_{\mu\nu} = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma}$. Covariant derivation of tensors is done via the Christoffel connection, $\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\rho}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} \right)$.

Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

follow from the Einstein-Hilbert action

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{|g|} R + S_{matter} \; .$$

Geodesics, i.e. trajectories of massive and massless particles, provide intuition about properties of a particular space.

Differential geometry gives a more general framework.
One introduces tangent space of vectors e_α, cotangent space of 1-forms θ^α; tensors are defined by tensor products.

Covariant derivative is given by a connection 1-form $\omega^\alpha{}_\beta$, which defines torsion and curvature as

$$\Theta^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta \,, \qquad \Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta \,.$$

When torsion vanishes and connection is metric-compatible, $\omega^\alpha{}_\beta$ reduces to the Christoffel connection.

Freely falling frame is a set of linearly independent orthonormal vector fields $\{e_\alpha\}$; elements the invariance group are the local Lorentz rotations of the set of ON frames, $e'_\alpha = \Lambda^\beta{}_\alpha e_\beta$.

This description perhaps describes better the principle of local invariance, PE'. Also, properties of space are more naturally expressed through behavior of fields (scalar, gauge, etc.) than point particles?

Gauge fields are also expressed in geometric language: the corresponding structure is that of a fiber bundle.
Fiber bundle is a manifold which is locally a direct product of a base space (in this case, spacetime) and a fiber (for principal bundle, a Lie group) + projection & properties.

In order to define paralell transport on the base manifold, one introduces connection $\omega \equiv A = A_\mu dx^\mu$, i.e. vector potential, and curvature, $\Omega \equiv F\, dA + A \wedge A$, equal to the field strength.

The Yang-Mills action is then

$$S_{YM} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int \Omega \wedge^* \Omega\ ,$$

quadratic in curvature, unlike the Einstein-Hilbert action.

What happens when we unify or connect the geometric descriptions of gravity and gauge fields?

Kaluza and Klein first proposed a unification of gravity with electrodynamics as a theory of gravity in 5d space (x^μ, y), where coordinate y belongs to a circle of small radius r. Assuming

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 A_\mu A_\nu & \kappa A_\mu \\ \kappa A_\nu & 1 \end{pmatrix} .$$

and the ‘cylinder condition’ that fields depend only on x^μ, the 5d Einstein-Hilbert action becomes

$$S = -\int d^4x \, \sqrt{|g|} \left(\frac{R}{16\pi G} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

Moreover, under transformations $y' = y + f(x), \; x' = x'(x)$, the vector potential transforms as $A'_\mu = A_\mu + \partial_\mu f$.

However, this simple model is not GR in 5d, as one cannot generally assume $\bar{g}_{55} = 1$, nor neglect the dependence on y.

Addition of a scalar field $\phi(x)$ gives a correct ansatz

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi^2 A_\mu A_\nu & \kappa\phi^2 A_\mu \\ \kappa\phi^2 A_\nu & \phi^2 \end{pmatrix}$$

i.e. the frame

$$\theta^\alpha = e^\alpha{}_\mu dx^\mu\,, \quad \theta^5 = \kappa A_\mu dx^\mu + \phi\, dy\ .$$

The implied 4d action

$$S = -\int d^4x\,\sqrt{|g|}\,\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\,\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{3}{2\kappa^2}\,\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi^2}\right)$$

gives more complicated equations of motion.

Dependence on y means expansion of fields in modes $e^{iny/r}$ and a discussion of vacuum vs. excited states of geometry.

The idea of KK to unify gravity with gauge theories was very influential in theoretical physics: it was revived after discovery of strong and weak forces, and extra dimensions were used to propose different unifying schemes and compactification scenarios in unified field theories, supergravity, string theory.

Complications which appear when extra dimensions describe nonabelian gauge theories can be circumvented by adding, to the Einstein-Hilbert Lagrangian, cosmological term, R^2, torsion, or matter. In any case, the infinite tower of heavy field modes remains, while some other problems can be solved.

Another realisation of the KK idea can be done within noncommutative geometry, where extra dimensions are finite matrix spaces.

Noncommutative geometry is ‘pointless’, that is, points x^μ are replaced by noncommutative objects $\hat{x}^\mu$ (matrices, operators). Scalar fields are functions of $\hat{x}^\mu$.

Noncommutative differential geometry with notions of derivative, differential, tensor; connection, curvature, Laplacian, etc. can be defined in several different ways.

These notions allow to define e.g. continuity on discrete spaces, like for example the space M_n of $n \times n$ matrices. It is clear that, whatever we take as a set of coordinates on this space, they will have discrete and finite spectra. One can also define the Laplacian and the Laplace equation: the scalar field then has a finite number of modes, upper-limited by n^2.

Let us introduce geometry of the M_n space in some detail. Denote by λ_a n^2-1 matrices which are generators of the $SU(n)$ group. As $\lambda_a\lambda_b = \frac{1}{2}\,C^c{}_{ab}\lambda_c + \frac{1}{2}\,d^c{}_{ab}\lambda_c - \frac{1}{n}\,g_{ab}$, they can be taken as coordinates (or better, momenta) in M_n.

The frame can be defined as follows:

$$e_a f = [\lambda_a, f]\,, \quad df = (e_a f)\theta^a = -[\theta, f]\,, \quad [f, \theta^a] = 0\,, \quad [e_a, e_b] = C^c{}_{ab} e_c\,,$$

where $\theta = -\lambda_a\theta^a$ is the 1-form, $d\theta + \theta\wedge\theta = 0$.

If we consider connections ω for $SU(n)$ transformations (given by unitary matrices g) and decomposition

$$\omega' = g^{-1}\omega g + g^{-1}dg = \theta' + \phi'$$

we find that $\theta' = \theta$ while the scalar ϕ transforms in the adjoint representation, $\phi' = g^{-1}\phi g$.

Using M_n and its geometry given above, one can define a noncommutative version of KK.

Define the extended spacetime as a product $V \otimes M_n$, where V is commutative, e.g. Minkowski space. Differential geometry is given by the frame $\{\theta^\alpha, \theta^a\}$, $\alpha = 0, 1, 2, 3$, $a = 1, \dots n^2 - 1$,

$$d = d_h + d_v = \theta^\alpha e_\alpha + \theta^a e_a \ .$$

The connection (not the frame, as before), decomposed as

$$\omega = \omega_h + \omega_v = A + \theta + \phi$$

gives, for elements of the curvature $\Omega = d\omega + \omega \wedge \omega$,

$$\Omega_{\alpha\beta} = F_{\alpha\beta} \ , \quad \Omega_{\alpha a} = D_\alpha \phi_a \ , \quad \Omega_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\phi_c$$

The corresponding Yang-Mills action is a sum of YM and Higgs action with quartic Higgs potential:

$$S = -\frac{1}{4}\mathrm{Tr}\left(\Omega_{\alpha\beta}\Omega^{\alpha\beta} + 2\Omega_{\alpha a}\Omega^{\alpha a} + \Omega_{ab}\Omega^{ab}\right) .$$

This example shows that noncommutative geometry gives additional possibilities for ‘discrete’ versions of the Kaluza-Klein mechanism. There are many models. A typical feature is that the extended space is a product of commutative and finite spaces, which gives finite number of KK modes.

The best known model is the Spectral Standard Model of Connes and Chamseddine, based on the spectral triple. Spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the algebra of functions, Hilbert space representation of spinors and the Dirac operator. A careful definition of the finite extension (and corresponding gauge+matter action) allows to introduce all fields of the SM.

Geometric spirit of this model is also in generalisation of the ‘reconstruction theorem’, which roughly says that, in the commutative case, knowing $(\mathcal{A}, \mathcal{H}, D)$ one can reconstruct properties of the initial manifold.

Our subgroup of group at the Faculty of Physics is not working, at present, on noncommutative particle physics models, but rather on noncommutative gravity or quantum spacetimes.

The idea is to quantise curved spacetimes themselves, representing coordinates x^μ by operators. The corresponding geometry is realised using the formalism mentioned before.

The latest results are on quantisation of the de Sitter and Anti-de Sitter spacetimes and analysis of equation of motion / propagators for real scalar fields on these fuzzy (A)dS spaces. A good property obtained is that, besides the metric, the scalar field modes and the propagator have the correct classical limit, that is, the correspondence principle is verified for fields too.

In dimensions $d > 2$, noncommutative structure introduces additional, ‘internal’ degrees of freedom.